On the homology of the double cobar construction of a double suspension

Alexandre Quesney

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Abstract

The double cobar construction of a double suspension comes with a Connes-Moscovici structure, that is a homotopy G-algebra (or Gerstenhaber-Voronov algebra) structure together with a particular BV-operator up to a homotopy. We show that the homology of the double cobar construction of a double suspension is a free BV-algebra. In characteristic two, a similar result holds for the underlying 2-restricted Gerstenhaber algebra. These facts rely on a formality theorem for the double cobar construction of a double suspension.

Introduction

For $n \geq 1$, the n-fold loop spaces (of $SO(n)$-spaces) are characterized by the (framed) little n-discs operad. In case $n = 1$, the cobar construction provides an algebraic model for the chain complex of loop spaces; it can be iterated to form the double cobar construction that provides a model for the chain complex of 2-fold loop spaces.

We are interested in further algebraic structures on the double cobar construction encoded by operads that are either equivalent to the chain little 2-discs operad, or equivalent to the chain *framed* little 2-discs operad in the case of 2-fold loop spaces of S^1-spaces. Recall that the homology operad of the little 2-discs operad is the Gerstenhaber operad and that the homology operad of the framed little 2-discs operad is the Batalin-Vilkovisky operad.

Let $\Omega^2 Y$ be the double loop space of a pointed topological space Y. Then its rational homology $H_*(\Omega^2 Y)$ is a Gerstenhaber algebra:

- the graded commutative product is the Pontryagin product $- \cdot -$;
- the degree 1 Lie bracket is the Browder bracket $\{-;-\}$;
- they are compatible in the following way:

  (0.1) $$\{a; b \cdot c\} = \{a; b\} \cdot c + (-1)^{(|a|+1)|b|} b \cdot \{a; c\},$$

  for all homogeneous $a, b, c \in H_*(\Omega^2 Y)$.

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If Y is endowed with an action of the circle S^1 (preserving the base point) then the double loop space $\Omega^2 Y = \mathrm{Hom}(S^2, Y)$ is also an S^1-space with the following action:

$$\begin{aligned}\Delta : S^1 \times \mathrm{Hom}(S^2, Y) &\to \mathrm{Hom}(S^2, Y)\\ (g, \phi) &\mapsto (g \cdot \phi)(s) := g \cdot \phi(g^{-1} \cdot s),\end{aligned}$$

where the circle S^1 acts on the 2-sphere S^2 by the base point-preserving rotations. This action induces (together with the Pontryagin product) a Batalin-Vilkovisky algebra structure $(H_*(\Omega^2 Y), \Delta)$, [Get94].

In [Que13] the author shows that the double cobar construction of a double suspension is endowed with a Connes-Moscovici structure. Essentially it is a homotopy G-algebra structure together with the boundary operator defined by A. Connes and H. Moscovici in [CM00], that is a BV-operator up to a homotopy. This can be seen as a first step for obtaining a strong homotopy BV-algebra structure on the double cobar construction.

Our main structural results are the followings.

Theorem (5.3)**.** *Let the coefficient field be* $\mathbb{Q}$*. Then* $\Omega^2 C_*(\Sigma^2 X)$ *has a Connes-Moscovici structure. The induced BV-algebra structure* $(H_*(\Omega^2 C_*(\Sigma^2 X)), \Delta_{CM})$ *is free on the reduced homology* $H_*^+(X)$*.*

Therefore the Batalin-Vilkovisky algebras $(H_*(\Omega^2 C_*(\Sigma^2 X)), \Delta_{CM})$ and $(H_*(\Omega^2 |\Sigma^2 X|), \Delta)$ are isomorphic.

In characteristic 2 the homology of double loop spaces is 2-restricted Gerstenhaber algebra (see Definition 3.2).We obtain

Theorem (4.1)**.** *Let the coefficient field be* $\mathbb{F}_2$*. Then* $\Omega^2 C_*(\Sigma^2 X)$ *has a Connes-Moscovici structure. The induced* 2*-restricted Gerstenhaber algebra structure on* $H_*(\Omega^2 C_*(\Sigma^2 X))$ *is free on the reduced homology* $H_*^+(X)$*.*

Therefore the restricted Gerstenhaber algebras $H_*(\Omega^2 C_*(\Sigma^2 X))$ and $H_*(\Omega^2 |\Sigma^2 X|)$ are isomorphic.

The two previous theorems use the following formality theorem.

Let us mention that what we call a homotopy BV-algebra cf.[Que13] has to be understood as a homotopy G-algebra with a BV-operator up to a homotopy. Such a structure has the Connes-Moscovici structure as an archetype.

Theorem (2.2)**.** *Let* $\Bbbk$ *be any coefficient field. Let* C *be a homotopy G-coalgebra with no higher structural co-operations and such that any element of* C^+ *is primitive for the coproduct. Then* $\Omega^2 C$ *and* $\Omega^2 H_*(C)$*, endowed with the Connes-Moscovici structure, are quasi-isomorphic as homotopy BV-algebras.*

The main example of such a C is the simplicial chain complex of a double suspension.

The paper is organized as follows.

In the first section: we fix notations, we recall both the cobar construction and what are homotopy G-(co)algebras. The main result is the Proposition 1.3, coming from [Que13], that claims the existence of the Connes-Moscovici structure on the double cobar construction of a double suspension. The second section is devoted to the formality theorem. The third section described the 2-restricted Gerstenhaber algebras coming from the homology of a homotopy G-algebra. In the fourth section we prove that, in characteristic two, the homology of the double cobar construction of a double suspension is a 2-restricted Gerstenhaber algebra. In the last section the coefficient field is $\mathbb{Q}$; we prove that the homology of the double cobar construction of a double suspension is BV-algebra. In the appendix we write the relations among the structural co-operations of homotopy G-coalgebras.

1 A few algebraic structures on the cobar construction

This section does not contain new results. Here we recall a few algebraic structures on both the cobar and the double cobar constructions. The main result of this section shows that the double cobar construction of an *involutive* homotopy G-coalgebra has a *Connes-Moscovici structure*. Roughly speaking, a Connes-Moscovici structure is a *homotopy G-algebra* structure together with a BV-operator up to a homotopy. The terminology Connes-Moscovici structure refers to the boundary operator defined on the cobar construction of an involutive Hopf algebra in [CM00]; it is here the BV-operator.

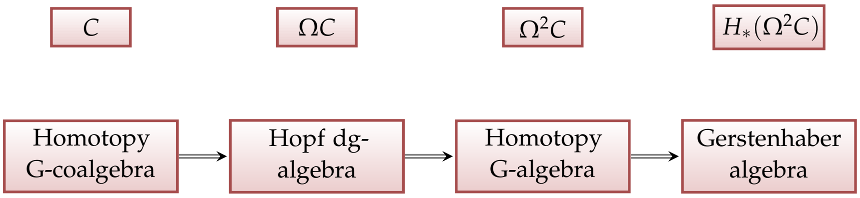
The cobar construction is a functor

$$\Omega : DGC \to DGA$$

from the category of 1-connected differential graded coalgebras to the category of connected differential graded algebras. By a 1-connected dg-coalgebra C we mean a co-augmented co-unital (of co-unit ϵ) dg-coalgebra such that $C_n = 0$ if $n < 0$ or $n = 1$ and $C_0 = \Bbbk$. As a vector space, the cobar construction of a dg-coalgebra $(C, \epsilon, d_C, \nabla_C)$ is defined as the free tensor algebra $T(\mathrm{s}^{-1} C^+)$. Here, s^{-1} is the usual desuspension, $(\mathrm{s}^{-1} M)_n = M_{n+1}$, and C^+ is the reduced dg-coalgebra $C^+ = \mathrm{Ker}(\epsilon)$. The differential d_Ω of the cobar construction ΩC is given as the unique derivation such that its restriction to $\mathrm{s}^{-1} C^+$ is

$$d_\Omega(\mathrm{s}^{-1} c) = -\mathrm{s}^{-1} d_{C^+}(c) + (\mathrm{s}^{-1} \otimes \mathrm{s}^{-1}) \nabla_{C^+}(c) \quad \forall c \in C^+.$$

The cobar construction of a dg-coalgebra C can be enriched with a Hopf dg-algebra structure for example when C is endowed with a *homotopy G-**co**algebra* structure, cf. [Kad04]. In this case the resulting double cobar construction $\Omega^2 C = \Omega(\Omega C)$ has a structure of homotopy G-algebra, cf. [Kad05].



A homotopy G-coalgebra structure on a dg-coalgebra C is the data of a coproduct $\nabla : \Omega C \to \Omega C \otimes \Omega C$ such that:

- $(\Omega C, \nabla)$ is a (co-unitary) dg-bialgebra;
- ∇ satisfies the left-sided condition (6.7) defined in the appendix.

The coproduct ∇ being dg-algebra morphism it corresponds to a unique twisting cochain $E : C^+ \to \Omega C \otimes \Omega C$. We note

$$E^{i,j} : C^+ \to (C^+)^{\otimes i} \otimes (C^+)^{\otimes j}$$

its (i, j)-component. The left-sided condition (6.7) says exactly that $E^{i,j} = 0$ when $i \geq 2$. Both the coassociativity of ∇ and the compatibility with the differential d_Ω lead to relations among the $E^{1,k}$'s. For its particular importance, we distinguish the co-operation $E^{1,1}$ and we denote it by ∇_1. In particular:

- ∇_1 is a chain homotopy for the cocommutativity of the coproduct of C;
- the coassociativity of ∇_1 is controlled by the co-operation $E^{1,2}$.

We refer to the appendix for the complete relationship among these co-operations.

Dually, a homotopy G-algebra structure on a dg-algebra A is the data of a product $\mu : \mathcal{B} A \otimes \mathcal{B} A \to \mathcal{B} A$ on the bar construction (see [Kad05]) such that:

- $(\mathcal{B} A, \mu)$ is a (unital) dg-bialgebra;
- μ satisfies a right-sided condition, see [LR10, 3.1 and 3.2].

The product μ being dg-coalgebra morphism it corresponds to a unique twisting cochain $E : \mathcal{B} A \otimes \mathcal{B} A \to A^+$. We note

$$E_{i,j} : (A^+)^{\otimes i} \otimes (A^+)^{\otimes j} \to A^+$$

its (i, j)-component. The right-sided condition [LR10, 3.1] says exactly that $E_{i,j} = 0$ when $i \geq 2$. The associativity of μ and the compatibility with the differential $d_\mathcal{B}$ of the bar construction leads to relations among the $E_{k,1}$'s. We distinguish the operation $E_{1,1}$ and we denote it by $\cup_1$. In particular:

- $\cup_1$ is a chain homotopy for the commutativity of the product of A;
- the associativity of $\cup_1$ is controlled by the operation $E_{1,2}$;

- $\cup_1$ is a left derivation for the product but it is a right derivation up to homotopy for the product.

We refer to [Que13] for explicit relations among the operations, or to [Kad05] in characteristic two. However, for later use, let us write precisely the three previous facts in characteristic two:

$$ab - ba = (\bar{\partial}\cup_1)(a \otimes b), \tag{1.1}$$

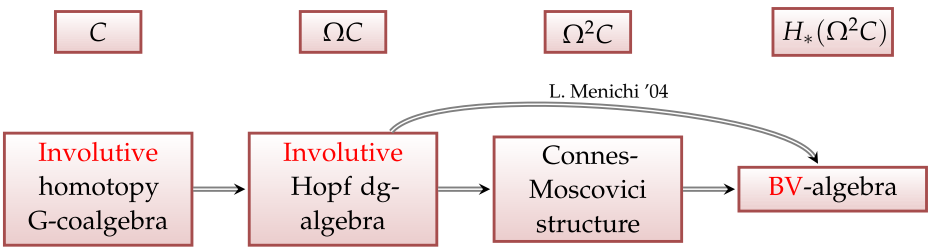
$$(a \cup_1 b) \cup_1 c - a \cup_1 (b \cup_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b), \tag{1.2}$$

$$ab \cup_1 c - a(b \cup_1 c) - (a \cup_1 c)b = 0, \tag{1.3}$$

$$a \cup_1 bc - (a \cup_1 b)c - b(a \cup_1 c) = \bar{\partial}E_{1,2}(a \otimes b \otimes c), \tag{1.4}$$

for all $a, b, c \in A^+$ and where, for $n \geq 1$, $\bar{\partial}(f) = d_A f - f d_{A^{\otimes n}}$ is the usual differential of $\mathrm{Hom}(A^{\otimes n}, A)$.

A. Connes and H. Moscovici [CM00, (2.21)] defined a boundary operator on the cobar construction of an involutive Hopf algebra (that is a Hopf algebra with an involutive antipode). This operator, named the Connes-Moscovici operator, induces a BV-algebra structure on the homology of such a cobar construction, [Men04]. In [Que13] the author showed that the cobar construction of an involutive Hopf dg-algebra $\mathcal{H}$ is endowed with the *Connes-Moscovici structure*. The latter lifts the BV-algebra structure obtained by L. Menichi [Men04] on $H_*(\Omega\mathcal{H})$. Moreover, the homotopy G-coalgebras such that the antipode on the resulting cobar construction is involutive was characterized as the *involutive* homotopy G-coalgebras.



Definition 1.1. Let $\mathcal{H}$ be an involutive Hopf dg-algebra $\mathcal{H}$. The Connes-Moscovici structure on the cobar construction $\Omega\mathcal{H}$ is the homotopy G-algebra structure $(\Omega\mathcal{H}, \cup_1, \{E_{1,k}\}_{k\geq 2})$ given in [Kad05] together with the Connes-Moscovici operator $\Delta_{CM} : \Omega\mathcal{H} \to \Omega\mathcal{H}$.

It satisfies the following properties:

$$\Delta_{CM}^2 = 0; \tag{1.5}$$

$$(-1)^{|a|}\big(\Delta_{CM}(ab) - \Delta_{CM}(a)b - (-1)^{|a|}a\Delta_{CM}(b)\big) = \{a; b\} + \bar{\partial}H(a; b), \tag{1.6}$$

for all homogeneous elements $a, b \in \Omega\mathcal{H}$, where H is a chain homotopy of degree 2 and $\{a; b\} := a \cup_1 b - (-1)^{(|a|+1)(|b|+1)} b \cup_1 a$.

The example of homotopy G-coalgebras C that we consider is the simplicial chain complex of a simplicial set.
Given a pointed simplicial set X, we denote by $C_*(X)$ the simplicial chain complex on X. The Alexander-Whitney coproduct makes $C_*(X)$ into a dg-coalgebra. We denote by $|X|$ a geometric realization of X.

H.-J. Baues [Bau81, Bau98] constructed a coproduct

$$\nabla_0 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$$

on the cobar construction $\Omega C_*(X)$ when the 1-skeleton of $|X|$ is a point. It satisfies:

- $(\Omega C_*(X), \nabla_0)$ is a Hopf dg-algebra;
- $H_*(\Omega^2 C_*(X)) \cong H_*(\Omega^2|X|)$ when the 2-skeleton of $|X|$ is a point.

This coproduct corresponds to a homotopy G-coalgebra structure on $C_*(X)$.

Definition 1.2. A homotopy G-coalgebra $(C, \nabla_1, \{E^{1,k}\}_{k\geq 2})$ is reduced if ∇_1 and $E^{1,k}$ for all $k \geq 2$ are zero.

Proposition 1.3. *[Que13, Theorem 4.6] Let $\Sigma^2 X$ be a double simplicial suspension. Then*

- *Baues' coproduct determines a reduced (hence involutive) homotopy G-coalgebra structure $C_*(\Sigma^2 X)$;*
- *the double cobar construction $\Omega^2 C_*(\Sigma^2 X)$ has the Connes-Moscovici structure.*

The Alexander-Whitney coproduct on $C_*(\Sigma^2 X)$ is primitive, that is all the elements of $C_*^+(\Sigma^2 X)$ are primitive for the coproduct. In fact this is already the case for a suspension ΣX.

2 Formality theorem

In order to prove a formality theorem for the double cobar construction endowed with the Connes-Moscovici structure we need to specify the category of what we call the homotopy BV-algebras.

Definition 2.1. A homotopy BV-algebra is a homotopy G-algebra $(A, \cup_1, \{E_{1,k}\}_{k\geq 2})$ with a degree 1 operator Δ satisfying:

(2.1) $\qquad \Delta^2 = 0;$

(2.2) $\qquad (-1)^{|a|}\big(\Delta(ab) - \Delta(a)b - (-1)^{|a|}a\Delta(b)\big) = \{a;b\} + \bar{\partial}H(a;b),$

for all homogeneous elements $a, b \in A$, where H is a chain homotopy of degree 2 and $\{a;b\} := a \cup_1 b - (-1)^{(|a|+1)(|b|+1)} b \cup_1 a$.
An ∞-morphism between two homotopy BV-algebras A and A' is defined as follows. It is:

- a morphism of dg-bialgebras $f : \mathcal{B}\, A \to \mathcal{B}\, A'$ between the associated bar constructions (i.e. an ∞-morphism of homotopy G-algebras), its components are $f_n : A^{\otimes n} \to A'$, $n \geq 1$; and,

- the linear map $f_1 : A \to A'$ commutes with the BV-operators, i.e. $f_1 \Delta = \Delta f_1$.

Here we prove that the double cobar construction of a reduced homotopy G-coalgebra is formal as homotopy BV-algebra in the following sense:

Theorem 2.2. *Let $\Bbbk$ be any coefficient field. Let C be a reduced (hence involutive) homotopy G-coalgebra with a coproduct such that all the elements of C^+ are primitive. Then $\Omega^2 C$ and $\Omega^2 H_*(C)$, endowed with the Connes-Moscovici structure, are quasi-isomorphic as homotopy BV-algebras.*

This theorem relies on the homotopy transfer theorem (HTT for short) for A_∞-coalgebras.

The latter transfers the A_∞-coalgebra structure along a contraction:

$$\begin{tikzcd} (C,d) \arrow[loop left, "\nu"] \arrow[rr, bend left, "p"] & & (H,d) \arrow[ll, bend left, "i"] \end{tikzcd}$$

where p and i are chain maps and ν is a chain homotopy such that:

$$\begin{aligned} pi &= Id \\ ip - Id &= d\nu + \nu d \\ p\nu = \nu i = \nu^2 &= 0 \ \text{ (Gauge conditions).} \end{aligned}$$

Theorem 2.3 (Transfer Theorem)**.** *[Gug82] Let (C, d, ∇) be a connected dg-coalgebra such that $C_1 = 0$, and (H, d) a dg-vector space. We suppose them related by a contraction as above. Then there exists on H an A_∞-coalgebra structure (H, ∂_i), and also an A_∞-quasi-isomorphism*

$$f : (C, d, \nabla) \to (H, \partial_i).$$

The ∂_i are given by:

$$\begin{aligned} \partial_0 &= d_H \\ \partial_1 &= (p \otimes p)\nabla i \\ \partial_2 &= p^{\otimes 3}(\nabla\nu \otimes Id - Id \otimes \nabla\nu)\nabla i, \end{aligned}$$

and more generally, for $i \geq 2$,

$$\partial_i = \sum_{\substack{0 \leq k_{i-1} \leq i-1 \\ 0 \leq k_{i-2} \leq i-2 \\ \dots \\ 0 \leq k_1 \leq 1}} \pm p^{\otimes i+1}(Id^{\otimes k_{i-1}} \otimes \nabla\nu \otimes Id^{\otimes i-1-k_{i-1}})(Id^{\otimes k} \otimes \nabla\nu \otimes Id^{\otimes i-1-k})\dots$$

$$\dots(\nabla\nu \otimes Id - Id \otimes \nabla\nu)\nabla i.$$

The quasi-isomorphism f is equivalent to a twisting cochain $\tau : C \to \Omega H$ with components $\tau = \sum_{i\geq 0} \tau_i$ where $\tau_i : C \to H^{\otimes i+1}$ is:

$$\begin{aligned}
\tau_0 &= p\\
\tau_1 &= (p\otimes p)\nabla\nu\\
\tau_2 &= p^{\otimes 3}(\nabla\nu\otimes Id - Id\otimes\nabla\nu)\nabla\nu
\end{aligned}$$

and more generally, for $i \geq 2$,

$$\tau_i = \sum_{\substack{0\leq k_{i-1}\leq i-1\\ 0\leq k_{i-2}\leq i-2\\ \cdots\\ 0\leq k_1\leq 1}} \pm p^{\otimes i+1}(Id^{\otimes k_{i-1}}\otimes\nabla\nu\otimes Id^{\otimes i-1-k_{i-1}})(Id^{\otimes k}\otimes\nabla\nu\otimes Id^{\otimes i-1-k})\ldots$$
$$\ldots(\nabla\nu\otimes Id - Id\otimes\nabla\nu)\nabla\nu.$$

Proof of Theorem 2.2. The chain complex C is formal as dg-coalgebra. More precisely, over a field we have a contraction (cf. [Gug82]),

$$\begin{tikzcd}
(C,d) \arrow[loop left, "\nu"] \arrow[rr, bend left, "p"] & & (H_*(C,d),0) \arrow[ll, bend left, "i"]
\end{tikzcd}$$

where, p and i are chain maps and ν is a chain homotopy such that:

$$pi = Id \tag{2.3}$$
$$ip - Id = d\nu + \nu d \tag{2.4}$$
$$p\nu = \nu i = \nu^2 = 0 \ \text{(Gauge conditions)}. \tag{2.5}$$

The transfer theorem for A_∞-coalgebra [Gug82] transfers the dg-coalgebra structure (seen as an A_∞-coalgebra) from C to $H_*(C,d)$. Since the coproduct of C is primitive, the obtained A_∞-coalgebra structure $(H_*(C,d),\delta_i)$ is reduced to the only coproduct

$$H(\nabla_C) : H_*(C,d) \to H_*(C,d)\otimes H_*(C,d).$$

Moreover, $H(\nabla_C)$ is primitive. Indeed, the HTT provides a differential (which is also a derivation) $\delta : \Omega H_*(C,d) \to \Omega H_*(C,d)$. This one is given by $\delta = \sum_{i\geq 1}\delta_i$ with $\delta_i(H_*(C,d)) \subset (H_*(C,d))^{\otimes i+1}$. Let us denote $H_*(C,d)$ by H. Explicitly, we have,

$$\begin{aligned}
\delta_0 &= d_H = 0\\
\delta_1 &= (p\otimes p)\nabla_C i = \nabla_H\\
\delta_2 &= p^{\otimes 3}(\nabla_C\nu\otimes Id - Id\otimes\nabla_C\nu)\nabla_C i,
\end{aligned}$$

and more generally, for $i \geq 2$,

$$\delta_i = \sum_{\substack{0\leq k_{i-1}\leq i-1\\ 0\leq k_{i-2}\leq i-2\\ \cdots\\ 0\leq k_1\leq 1}} \pm p^{\otimes i+1}(Id^{\otimes k_{i-1}} \otimes \nabla_C \nu \otimes Id^{\otimes i-1-k_{i-1}})(Id^{\otimes k} \otimes \nabla_C \nu \otimes Id^{\otimes i-1-k})...$$

$$...(\nabla_C \nu \otimes Id - Id \otimes \nabla_C \nu)\nabla_C i.$$

In our case, ∇_C is primitive, and since ν satisfies the gauge conditions, then only δ_1 is not zero. Moreover, let us write the twisting cochain $\tau : C \to \Omega H$ provided by the HTT. Its components $\tau = \sum_{i\geq 0} \tau_i$ where $\tau_i : C \to H^{\otimes i+1}$ are:

$$\begin{aligned}
\tau_0 &= p\\
\tau_1 &= (p \otimes p)\nabla_C \nu\\
\tau_2 &= p^{\otimes 3}(\nabla_C \nu \otimes Id - Id \otimes \nabla_C \nu)\nabla_C \nu
\end{aligned}$$

and more generally, for $i \geq 2$,

$$\tau_i = \sum_{\substack{0\leq k_{i-1}\leq i-1\\ 0\leq k_{i-2}\leq i-2\\ \cdots\\ 0\leq k_1\leq 1}} \pm p^{\otimes i+1}(Id^{\otimes k_{i-1}} \otimes \nabla_C \nu \otimes Id^{\otimes i-1-k_{i-1}})(Id^{\otimes k} \otimes \nabla_C \nu \otimes Id^{\otimes i-1-k})...$$

$$...(\nabla_C \nu \otimes Id - Id \otimes \nabla_C \nu)\nabla_C \nu$$

Thus it is reduced to

$$\tau_0 = p : C \to H.$$

Thus, p is an A_∞-coalgebra morphism, i.e. Ωp is dg-algebra morphism. Its components $(\Omega p)_n$ on $(\mathrm{s}^{-1}\, C)^{\otimes n}$ are given by:

$$(\Omega p)_n = \mathrm{s}^{-1}\, p\, \mathrm{s} \otimes \mathrm{s}^{-1}\, p\, \mathrm{s} \otimes \cdots \otimes \mathrm{s}^{-1}\, p\, \mathrm{s}\,.$$

We obtain the contraction

$$\begin{tikzcd}
(\Omega C, d) \arrow[loop left, "\Gamma"] \arrow[rr, bend left, "\Omega p"] & & (\Omega H, 0) \arrow[ll, bend left, "\Omega i"]
\end{tikzcd}$$

where,

$$\Omega p \Omega i = Id \tag{2.6}$$

$$\Omega i \Omega p - Id = d\Gamma + \Gamma d \tag{2.7}$$

$$\Omega p \Gamma = \Gamma \Omega i = \Gamma^2 = 0 \ \text{(Gauge conditions)}, \tag{2.8}$$

and where Γ is defined by its restrictions Γ_n to $(\mathrm{s}^{-1}\, C)^{\otimes n}$:

$$(\Gamma)_n = \sum_{\substack{k_1+\cdots+k_{s+1}=n-s\\ 1\leq s\leq n}} \pm Id^{\otimes k_1} \otimes (\mathrm{s}^{-1}\, \nu\, \mathrm{s}) \otimes Id^{\otimes k_2} \otimes (\mathrm{s}^{-1}\, \overline{\partial} \nu\, \mathrm{s}) \otimes Id^{\otimes k_3} \otimes (\mathrm{s}^{-1}\, \overline{\partial} \nu\, \mathrm{s}) \otimes Id^{\otimes k_4}$$

$$\otimes \cdots \otimes (\mathrm{s}^{-1}\, \overline{\partial} \nu\, \mathrm{s}) \otimes Id^{\otimes k_{s+1}},$$

where $\overline{\partial}\nu := d\nu + \nu d$. Equivalently, setting $\Gamma_1 := \mathrm{s}^{-1}\,\nu\,\mathrm{s}$, we have:

$$\Gamma_n = Id \otimes \Gamma_{n-1} \pm \mathrm{s}^{-1}\,\nu\,\mathrm{s} \otimes \overline{\partial}\Gamma_{n-1} \pm \mathrm{s}^{-1}\,\nu\,\mathrm{s} \otimes Id^{\otimes n-1},\ n \geq 2.$$

The homotopy G-coalgebra structure of C endowed ΩC with a dg-coalgebra structure; we set $(\Omega C, d, \nabla_0)$ the obtained dg-coalgebra. This dg-coalgebra structure on ΩC, seen as A_∞-coalgebra structure, transfers to an A_∞-coalgebra structure on ΩH, which is A_∞-equivalent to the dg-coalgebra $(\Omega C, d, \nabla_0)$. Let us show that the obtained A_∞-coalgebra structure $(\Omega H_*(\Sigma^2 X), \partial_i)$ is reduced to the coproduct $H_*(\nabla_0)$. The higher coproducts

$$\partial_i : \Omega H \to (\Omega H)^{\otimes i+1},$$

are given by:

$$\begin{aligned}
\partial_0 &= 0\\
\partial_1 &= (\Omega p \otimes \Omega p)\nabla_0 \Omega i = H_*(\nabla_0)\\
\partial_2 &= (\Omega p)^{\otimes 3}(\nabla_0\Gamma \otimes Id - Id \otimes \nabla_0\Gamma)\nabla_0\Omega i
\end{aligned}$$

and more generally, for $i \geq 2$,

$$\partial_i = \sum_{\substack{0\leq k_{i-1}\leq i-1\\ 0\leq k_{i-2}\leq i-2\\ \cdots\\ 0\leq k_1\leq 1}} \pm(\Omega p)^{\otimes i+1}(Id^{\otimes k_{i-1}} \otimes \nabla_0\Gamma \otimes Id^{\otimes i-1-k_{i-1}})(Id^{\otimes k} \otimes \nabla_0\Gamma \otimes Id^{\otimes i-1-k})\ldots$$

$$\ldots(\nabla_0\Gamma \otimes Id - Id \otimes \nabla_0\Gamma)\nabla_0\Omega i.$$

Let us remark that: each term of Γ has at least one operation ν; the coproduct ∇_0, being a shuffle-coproduct, is the composition of deconcatenations followed by permutations. Thus, each term of $\nabla_0\Gamma$ has the operation ν. Then, because of the gauge condition $p\nu = 0$, the post-composition by Ωp vanishes each term of $\nabla_0\Gamma$.
Therefore only the co-operation $H_*(\nabla_0)$ is non trivial.

The same argument shows that the A_∞-quasi-isomorphism between the dg-coalgebras $(\Omega C, d, \nabla_0)$ and $(\Omega H, 0, H_*(\nabla_0))$ is strict. Indeed, the twisting cochain

$$\tau : \Omega C \to \Omega^2 H,$$

with components $\tau_i : \Omega C \to (\Omega H)^{\otimes i+1}$ is:

$$\begin{aligned}
\tau_0 &= \Omega p\\
\tau_1 &= (\Omega p \otimes \Omega p)\nabla_0\Gamma\\
\tau_2 &= (\Omega p)^{\otimes 3}(\nabla_0\Gamma \otimes Id - Id \otimes \nabla_0\Gamma)\nabla_0\Gamma,
\end{aligned}$$

and more generally, for $i \geq 2$,

$$\tau_i = \sum_{\substack{0\leq k_{i-1}\leq i-1\\ 0\leq k_{i-2}\leq i-2\\ \cdots\\ 0\leq k_1\leq 1}} \pm(\Omega p)^{\otimes i+1}(Id^{\otimes k_{i-1}} \otimes \nabla_0\Gamma \otimes Id^{\otimes i-1-k_{i-1}})(Id^{\otimes k} \otimes \nabla_0\Gamma \otimes Id^{\otimes i-1-k})\ldots$$

$$\ldots(\nabla_0\Gamma \otimes Id - Id \otimes \nabla_0\Gamma)\nabla_0\Gamma.$$

As a consequence, the double cobar construction $\Omega^2 C$ is quasi-isomorphic to $\Omega^2 H$, as homotopy BV-algebras. $\square$

3 The 2-restricted G-algebras

In this section the field of coefficients is $\mathbb{F}_2$.

Definition 3.1. Let $r \in \mathbb{N}$. A 2-restricted Lie algebra of degree r is a connected graded vector space L endowed with a Lie bracket of degree r,

$$[-;-]_r : L_j \otimes L_k \to L_{j+k+r}, \quad j,k \geq 0,$$

and a restriction,

$$\xi_r : L_n \to L_{2n+r},$$

satisfying:

1. $[-;-]_r$ is symmetric;
2. $\xi_r(kx) = k^2 \xi_r(x)$;
3. $[x;[y;z]_1]_r + [y;[z;x]_r]_r + [z;[x;y]_r]_r = 0$;
4. $[\xi(x);y]_r = [x;[x;y]_r]_r$; and,
5. $\xi_r(x+y) = \xi_r(x) + [x;y]_1 + \xi_r(y)$,

for all homogeneous elements $x,y,z \in L$.

Definition 3.2. A 2-restricted Gerstenhaber algebra is a degree one 2-restricted Lie algebra, G, and a graded symmetric algebra such that:

1. $[x,yz]_1 = [x,y]_1 z + y[x,z]_1$;
2. $\xi_1(xy) = x^2 \xi_1(y) + \xi_1(x) y^2 + x[x;y]_1 y$,

for all homogeneous elements $x,y,z \in G$.

Proposition 3.3. *The homology $H_*(A,d)$ of a homotopy G-algebra $(A,d,\cdot,\cup_1,\{E_{1,k}\}_{k\geq 2})$ over $\mathbb{F}_2$ is a 2-restricted Gerstenhaber algebra. The restriction is induced by the map $x \mapsto x \cup_1 x$.*

Remark 3.4. The restriction ξ_1 is a part of Dyer-Lashof operations ξ_1, ζ_1 [Coh76] in prime characteristic. In [Tou06] V. Turchin gives explicit formulas for the operations in homotopy G-algebras inducing the Dyer-Lashof operations in homology. He showed via a different method that, in prime characteristic p, a homotopy G-algebra A is, in particular, a p-restricted Lie algebra of degree 1.

Proof. The bracket $[x;y]_1 = x \cup_1 y + y \cup_1 x$ together with the multiplication of A define a Gerstenhaber algebra structure on $H_*(A)$, [Kad05]. We set,

$$\xi_1(x) := x \cup_1 x,$$

for all $x \in A$. To alleviate notation, let us remove temporarily the index 1 and write ξ_1 as ξ. It suffices to check that ξ induces a homological map satisfying the properties of a restriction. First, we show the properties 4 and 5 from Definition 3.1 are satisfied on A, next, we show that the property 2 from Definition 3.2 is satisfied up to homotopy.

We show the equality $[\xi(x);y]_1 = [x;[x;y]_1]_1$. We have

(3.1)
$$\begin{aligned}[x;[x;y]_1]_1 &= x \cup_1 (x \cup_1 y + y \cup_1 x) + (x \cup_1 y + y \cup_1 x) \cup_1 x \\ &= x \cup_1 (x \cup_1 y) + x \cup_1 (y \cup_1 x) + (x \cup_1 y) \cup_1 x + (y \cup_1 x) \cup_1 x.\end{aligned}$$

From the equation (1.2), the two first terms satisfy:

$$\begin{aligned}x \cup_1 (x \cup_1 y) &= (x \cup_1 x) \cup_1 y + E_{1,2}(x;x,y) + E_{1,2}(x;y,x) \ ; \\ x \cup_1 (y \cup_1 x) &= (x \cup_1 y) \cup_1 x + E_{1,2}(x;y,x) + E_{1,2}(x;x,y).\end{aligned}$$

Then

$$x \cup_1 (x \cup_1 y) + x \cup_1 (y \cup_1 x) = (x \cup_1 x) \cup_1 y + (x \cup_1 y) \cup_1 x.$$

Similarly, the last term from (3.1) reads:

$$(y \cup_1 x) \cup_1 x = y \cup_1 (x \cup_1 x) + E_{1,2}(y;x,x) + E_{1,2}(y;x,x) = y \cup_1 (x \cup_1 x).$$

Thus,

$$[x;[x;y]_1]_1 = (x \cup_1 x) \cup_1 y + y \cup_1 (x \cup_1 x) = \xi(x) \cup_1 y + y \cup_1 \xi(x) = [\xi(x);y]_1.$$

The following equality $\xi(x+y) = \xi(x) + [x;y]_1 + \xi(y)$ is straightforward:

(3.2)
$$\begin{aligned}\xi(x+y) &= (x+y) \cup_1 (x+y) \\ &= x \cup_1 x + y \cup_1 x + x \cup_1 y + y \cup_1 y \\ &= \xi(x) + [x;y]_1 + \xi(y).\end{aligned}$$

The equality $\xi(xy) = x^2\xi(y) + \xi(x)y^2 + x[x;y]_1 y$ is satisfied up to homotopy on the chain complex A. This is a consequence of the equations (1.3) and (1.4). Denoting by $\sim$ the homotopy relation, we can write,

$$\begin{aligned}\xi(xy) &= xy \cup_1 xy \\ &= x(y \cup_1 xy) + (x \cup_1 xy)y \\ &\sim x^2(y \cup_1 y) + x(y \cup_1 x)y + x(x \cup_1 y)y + (x \cup_1 x)y^2 \\ &\sim x^2\xi(y) + x[x;y]_1 y + \xi(x)y^2\end{aligned}$$

Now we show that ξ induces a restriction on the homology $H_*(A)$.
From the equation (1.1) the operation $\cup_1$ is a homotopy for the commutativity of the product. In particular,

$$d(x \cup_1 x) + dx \cup_1 x + x \cup_1 dx = xx + xx = 0, \text{ for all } x \in A.$$

Thus, if $x \in A$ is a cycle,

$$d\xi(x) = d(x \cup_1 x) = dx \cup_1 x + x \cup_1 dx = 0 + 0 = 0.$$

If $y = dx \in A$ is a boundary, then

$$\xi(y) = y \cup_1 y = dx \cup_1 dx = d(dx \cup_1 x) + d^2x \cup_1 x + dx \cdot x + xdx = d(dx \cup_1 x) + d(x^2),$$

is a boundary. Thus, for a cycle $x \in A$ and an element $y \in A$, we obtain from (3.2):

$$\begin{aligned} \xi(x + dy) &= \xi(x) + [x; dy]_1 + \xi(dy) \\ &= \xi(x) + d[x; y]_1 + [dx; y]_1 + d(dy \cup_1 y) \\ &= \xi(x) + d[x; y]_1 + d(dy \cup_1 y) + d(y^2). \end{aligned}$$

Therefore ξ induces a restriction in homology. □

Remark 3.5. The above proposition also results from the following fact. Recall that for a homotopy G-algebra A the bar construction $\mathcal{B}\,A$ is a dg-bialgebra. Then, by composition with the restriction morphism $\mathcal{L}_p \to \mathcal{AS}$, the bar construction $\mathcal{B}\,A$ is a p-restricted Lie algebra (of degree 0). The projections on A of both the bracket and the restriction of $\mathcal{B}\,A$ correspond respectively to the bracket and the restriction described in [Tou06]. The latter operations make A into a p-restricted Lie algebra of degree 1. In other words, the Lie bracket $[-;-]_0$ and the restriction ξ_0 on $\mathcal{B}\,A$ give, after projection, the degree one Lie bracket $[-;-]_1$ and the restriction ξ_1 on A, respectively.

Definition 3.6. A morphism of 2-restricted Gerstenhaber algebras is a degree 0 linear map compatible with the product, the bracket and the restriction.

Definition 3.7. An ∞-morphism of homotopy G-algebras A and A' is a morphism of unital dg-bialgebras $f : \mathcal{B}\,A \to \mathcal{B}\,A'$.

Such a morphism is a collection of maps $f_n : A^{\otimes n} \to A'$, $n \geq 1$ satisfying some relations, see [Que13, Appendix Definition 4.17] for details. In particular,

$$\bar{\partial} f_1 = 0; \tag{3.3}$$

$$f_1(x) \cup_1 f_1(y) + f_1(x \cup_1 y) = f_2(x; y) + f_2(y; x) \text{ for all } x, y \in A. \tag{3.4}$$

An ∞-morphism of homotopy G-algebras induces a morphism of Gerstenhaber algebras in homology, see [Que13, Appendix, Proposition 4.20]. A direct check shows the compatibility of the ∞-morphisms with the restriction:

Proposition 3.8. *An ∞-morphism of homotopy G-algebras induces a morphism of 2-restricted Gerstenhaber algebras in homology.*

Proof. Take $y = x$ in the equation (3.4), this gives:

$$f_1(\xi_1(x)) = \xi_1(f_1(x)).$$

□

Now we define the free 2-restricted Gerstenhaber algebra.

Let L_{1r} be a 2-restricted Lie algebra of degree 1. Then the free graded commutative algebra on L_{1r}, $S(L_{1r})$ is a 2-restricted Gerstenhaber algebra. Indeed, the Poisson identity

$$[xy;z]_1 = x[y;z]_1 + [x;z]_1 y,$$

is equivalent to the fact that

$$\begin{aligned} ad_1(x) := [-;x]_1 : S(L_{1r}) &\to S(L_{1r}) \\ y &\mapsto [y;x]_1 \end{aligned}$$

is a derivation of degree $+1$ for the algebra $(S(L_{1r}), \cdot)$. Let us consider a basis of L_{1r} as k-vector space. It is sufficient to extend the application $ad_1(l)$: $L_{1r} \to L_{1r} \subset S(L_{1r})$ as a derivation on $S(L_{1r})$ for all $l \in L_{1r}$, and next to extend $[x;-]_1 : L_{1r} \to S(L_{1r})$ for all $x \in S(L_{1r})$.
The restriction is extended via the property 2 from Definition 3.2.

Let V be a dg-vector space with $V_0 = 0$. If $L_{1r}(V)$ denotes the free degree one 2-restricted Lie algebra on V, then $S(L_{1r}(V))$ is the free 2-restricted Gerstenhaber algebra on V.

4 The homology of the double cobar construction of a double suspension over $\mathbb{F}_2$

Theorem 4.1. *Let the coefficient field be $\mathbb{F}_2$. Then $\Omega^2 C_*(\Sigma^2 X)$ has a Connes-Moscovici structure. The induced 2-restricted Gerstenhaber algebra structure on $H_*(\Omega^2 C_*(\Sigma^2 X))$ is free on the reduced homology $H_*^+(X)$.*

It was proved in [Coh76] that $H_*(\Omega^2\Sigma^2 Y)$ is the free 2-restricted Gerstenhaber algebra on $H_*^+(Y)$, then we obtain

Corollary 4.2. *The restricted Gerstenhaber algebras $H_*(\Omega^2 C_*(\Sigma^2 X))$ and $H_*(\Omega^2|\Sigma^2 X|)$ are isomorphic.*

Proof of Theorem 4.1. From Proposition 1.3 and the formality Theorem 2.2 the homotopy G-algebras $\Omega^2 C_*(\Sigma^2 X)$ and $\Omega^2 H_*(\Sigma^2 X)$ are quasi-isomorphic. Therefore, it remains to prove that the structure of homotopy G-algebra on the double cobar construction $\Omega^2 H_*(\Sigma^2 X)$ induces the free 2-restricted Gerstenhaber algebra on $H_*^+(X)$.

We denote by $[-;-]_1$ the Gerstenhaber bracket on $H_*(\Omega^2 H_*(\Sigma^2 X))$,

$$[x;y]_1 = x \cup_1 y + y \cup_1 x,$$

and we denote by ξ_1 the restriction,

$$\xi_1(x) = x \cup_1 x,$$

for all $x \in H_*(\Omega^2 H_*(\Sigma^2 X))$.

We have an inclusion $\iota : H_*^+(X) \cong s^{-2} H_*^+(\Sigma^2 X) \hookrightarrow H_*(\Omega^2 H_*(\Sigma^2 X))$. Thus, from the freeness of the functors S and L_{1r}, we have the commutative diagram:

$$\begin{tikzcd} SL_{1r}(H_*^+(X)) \arrow[r, "SL_{1r}(\iota)"] & H_*(\Omega^2 H_*(\Sigma^2 X)) \\ H_*^+(X) \arrow[u, "i"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

where $SL_{1r}(\iota)$ is a morphism of 2-restricted Gerstenhaber algebras. It remains to show that $SL_{1r}(\iota)$ is an isomorphism. To do this, we construct two spectral sequences $\overline{E}$ and E, and a morphism $\Phi : E \to \overline{E}$ of spectral sequences satisfying:

$$TH_*^+(\Sigma X) \otimes SL_{1r}H_*^+(X) = E^2 \xrightarrow{T(\iota) \otimes S(L_{1r}(\iota))} \overline{E}^2 = H_*(\Omega H_*(\Sigma^2 X)) \otimes H_*(\Omega^2 H_*(\Sigma^2 X)).$$

We conclude by mean of the comparison theorem [ML95, Theorem 11.1 p.355], using that both $T(\iota)$ and Φ^∞ are isomorphisms.

Let us consider the application

$$\pi : \Omega H_*(\Sigma^2 X) \otimes \Omega^2 H_*(\Sigma^2 X) \to \Omega H_*(\Sigma^2 X)$$

defined by $\pi(x \otimes y) = \epsilon(y)x$, where ϵ is the augmentation of $\Omega H_*(\Sigma^2 X)$. The total space

$$\mathcal{E} := \Omega H_*(\Sigma^2 X) \otimes \Omega^2 H_*(\Sigma^2 X)$$

is endowed with a differential d and a contracting homotopy s defined as in [AH56, Axioms (R1),(R2),(D1),(D2)], see below for the details. The total space is acyclic [AH56, Lemme 2.3]. We set

$$B = \Omega H_*(\Sigma^2 X)$$

and

$$A = \Omega^2 H_*(\Sigma^2 X)$$

so that $\mathcal{E} = B \otimes A$. We define a filtration of $\mathcal{E}$ as

$$F_r\mathcal{E} = \bigoplus_{p \leq r} B_p \otimes A.$$

Following [AH56], we get a spectral sequence $\overline{E}^r$ associated to this filtration. The abutment is the homology of $\mathcal{E}$, the second page is

$$\overline{E}^2_{p,q} = H_p(\Omega H_*(\Sigma^2 X)) \otimes H_q(\Omega^2 H_*(\Sigma^2 X)).$$

The total space being acyclic we have $\overline{E}^\infty_{0,0} = \mathbb{F}_2$, $\overline{E}^\infty_{p,q} = 0$ if $(p,q) \neq (0,0)$.
The following spectral sequence of dg-algebras is used by F. Cohen [Coh76, III,p.228] as a model for the Serre spectral sequence associated to the paths fibration,

$$\Omega^{n+1}\Sigma^{n+1}X \to P\Omega^n\Sigma^{n+1}X \to \Omega^n\Sigma^{n+1}X, \;\; n > 0.$$

He shows that the homology $H_*(\Omega^{n+1}\Sigma^{n+1}X)$ is the free 2-restricted Gerstenhaber algebra of degree n on $H_*^+(X)$.

We defined the spectral sequence E^r. Its second page is given by

$$E^2_{p,q} = (T(H_*^+(\Sigma X)))_p \otimes (S(L_{1r}(H_*^+(X))))_q,$$

the differentials are defined such that E^r is a spectral sequence of dg-algebras whose transgressions are

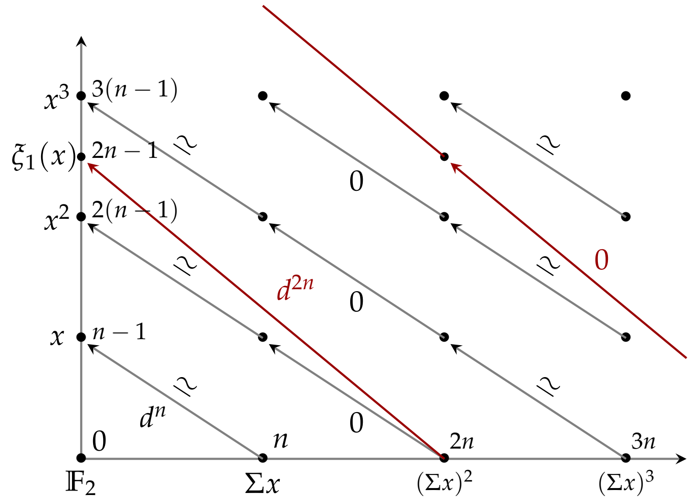
$$\begin{aligned} t(\Sigma x) &= x \\ t(\xi_0^k(\Sigma x)) &= \xi_1^k(x) \\ t(ad_0(\Sigma x_1)\cdots ad_0(\Sigma x_{k-1})(\Sigma x_k)) &= ad_1(x_1)\cdots ad_1(x_{k-1})(x_k), \end{aligned}$$

for all $\Sigma x, \Sigma x_1, ..., \Sigma x_k \in H_*^+(\Sigma X)$, where:

- $\xi_0(x) = x^2$ is the restriction of $TH_*^+(\Sigma X)$;
- ξ_1 is the restriction of $SL_{1r}(H_*^+(X))$;
- $[-;-]_0$ is the Lie bracket of $TH_*^+(\Sigma X)$;
- $[-;-]_1$ is the Gerstenhaber bracket of $SL_{1r}H_*^+(X)$; and,
- $ad_j(x)(y) = [y;x]_j$ for $j = 0$ or 1.

Then the r-th transgression $t = d^r : E^r_{r,0} \to E^r_{0,r-1}$ is extended to $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ as a derivation.

Let us investigate the example of one generator $\Sigma x \in H_n^+(\Sigma X)$ of degree n; the spectral sequence is:



All the brackets are zero since $[\Sigma x; \Sigma x]_0 = 2(\Sigma x)^2 = 0$ and $[\xi_0(\Sigma x); \Sigma x]_0 = [\Sigma x; [\Sigma x; \Sigma x]_0]_0$; it remains: the element Σx, the restriction $\xi_0(\Sigma x) = (\Sigma x)^2$ and its iterations, plus the resulting products. This spectral sequence split into elementary spectral sequences on the transgressive elements Σx, $\xi_0(\Sigma x)$, $\xi_0^2(\Sigma x)$,... with differentials the corresponding transgression. Indeed, $T(\Sigma x)$ can be written additively as the exterior algebra on the transgressive elements $\Sigma x, \xi_0^k(\Sigma x)$, $k \geq 1$. Also, $S(L_{1r}(x))$ can be written additively as polynomial algebra on the image of these transgressive elements $x, \xi_1^k(x)$, $k \geq 1$. Then the elementary spectral sequences are of the form

$$\Lambda(y) \otimes P(t(y)),$$

where y run through the transgressive elements of $T(\Sigma x)$. Such a presentation of $T(\Sigma x)$ can be extended to $TH_*^+(\Sigma X)$ via the $[-;-]_0$*-elementary products*. This presentation is due to P. J. Hilton [Hil55]. Let us consider the Lie algebra $L(H_*^+(\Sigma X))$ without restriction, and with bracket $[-;-]_0$. The $[-;-]_0$-elementary products are defined as follows.
The elements $x \in H_*^+(\Sigma X)$ are the $[-;-]_0$-elementary product of weight 1. Suppose that the $[-;-]_0$-elementary products of weight j have been defined for $j < k$. Then, a product of weight k is an element $[x;y]_0$ such that:

1. x is a $[-;-]_0$-elementary products of weight u;
2. y is a $[-;-]_0$-elementary products of weight v;
3. $u + v = k$;
4. $x < y$, and if $y = [z;t]_0$ for z and t elementary, then $z \leq x$.

We order the generators $(\Sigma x_i)_i$ of $H_*^+(\Sigma X)$ by their dimension; the $[-;-]_0$-elementary products are the elements

$$\Sigma x_k, \cdots, [\Sigma x_k; \Sigma x_l]_0, \cdots, [\Sigma x_j; [\Sigma x_k; \Sigma x_l]_0]_0, \cdots$$

where $j < k < l$. From [Hil55] we obtain

$$L(H_*^+(\Sigma X)) \cong \text{span}\{y \mid y \text{ a } [-;-]_0 - \text{elementary product}\},$$

as vector spaces. Denoting by p_j the $[-;-]_0$-elementary products we have the isomorphism of vector spaces

$$TH_*^+(\Sigma X) \cong \bigotimes_j P(p_j),$$

where P denotes the polynomial algebra. To take into account the restriction, we write $P(p_j)$ as $\wedge_{k\geq 0}(p_j^{2^k})$ where $\wedge$ denotes the exterior algebra[1]. We obtain a decomposition into $[-;-]_0$-elementary products and ξ_0-*elementary products*,

$$TH_*^+(\Sigma X) \cong \bigotimes_j \bigwedge_{k\geq 0} (p_j^{2^k}).$$

On the other hand, we recall that $L_{1r} = \mathrm{s}^{-1} L_r \mathrm{s}$, that is $[x;y]_1 = \mathrm{s}^{-1}[\mathrm{s}\, x; \mathrm{s}\, y]_0$ and $\xi_1(x) = \mathrm{s}^{-1} \xi_0(\mathrm{s}\, x)$. Then we obtain that

$$SL_{1r}(H_*^+(\Sigma X)) \cong S(\text{span}\{y, \xi_1^k(y) \mid y \text{ is a } [-;-]_1 - \text{elementary product, } k \geq 1\}).$$

Then the spectral sequence E decomposes into tensor products of elementary spectral sequences

$$\wedge(y) \otimes P(t(y)),$$

where y run through the set of $[-;-]_0$-elementary products and ξ_0-elementary products.

Thus the spectral sequence E converges to $E_{0,0}^\infty = \mathbb{F}_2$, $E_{p,q}^\infty = 0$ if $(p,q) \neq (0,0)$.

Return to the application

$$T(\iota) \otimes S(L_{1r}(\iota)) : E^2 \to \overline{E}^2.$$

From the comparison theorem [ML95, Theorem 11.1 p. 355], it is sufficient to show that: the application $\Phi^2 := T(\iota) \otimes S(L_{1r}(\iota))$ induces morphism of spectral sequences $\Phi : E \to \overline{E}$; the morphism $T(\iota)$ is an isomorphism; and $\Phi^\infty : E^\infty \to \overline{E}^\infty$ is an isomorphism.

We show that the application Φ^2:

$$TH_*^+(\Sigma X) \otimes SL_{1r}H_*^+(X) \xrightarrow{T(\iota)\otimes S(L_{1r}(\iota))} H_*(\Omega H_*(\Sigma^2 X)) \otimes H_*(\Omega^2 H_*(\Sigma^2 X))$$

[1] To avoid any ambiguity, we recall that over $\mathbb{F}_2$, the exterior algebra $\wedge V$ of a vector space V is defined as the tensor algebra TV quotiented by the ideal generated by $\{v \otimes v \mid v \in TV\}$.

induces a morphism of spectral sequences.

To do this, let us write with details the differential of $\mathcal{E} = B \otimes A$. The axioms (R1) and (R2) determine a contracting homotopy on $\mathcal{E}$:

(R1) $\quad s(1) = 0,\ s(\mathrm{s}^{-1}x) = x,\ s(x) = 0, \qquad$ for $\mathrm{s}^{-1}x \in A$

(R2) $\quad s(xy) = s(x)y + \epsilon(x)s(y), \qquad$ for $x \in \mathcal{E}, y \in A.$

The axioms (D1) and (D2) determine he differential of $\mathcal{E}$:

(D1) $\quad d(x) = (1 - sd)(\mathrm{s}^{-1}x), \qquad$ for $x \in B$

(D2) $\quad d(xy) = d(x)y + (-1)^{|x|}xd(y), \qquad$ for $x \in \mathcal{E}, y \in A.$

Then, for all $x \in \mathrm{s}^{-1} H^+_{p+1}(\Sigma^2 X) \subset B_p$, we obtain

$$d(x) = \mathrm{s}^{-1}x \in \mathrm{s}^{-2} H^+_{p+1}(\Sigma^2 X) \subset F_0\mathcal{E},$$

since the coproduct ∇_0 is primitive on $H^+_*(\Sigma^2 X)$. Thus, for such elements x of degree p, we have $d^r(x) = 0$ if $r < p$. Moreover, for all $x \in \mathrm{s}^{-1} H^+_{p+1}(\Sigma^2 X) \subset \overline{E}^p_{p,0} \subset B_p$, we obtain

$$d^p(x) = \mathrm{s}^{-1}x \in \mathrm{s}^{-2} H^+_{p+1}(\Sigma^2 X) \subset \overline{E}^p_{0,p-1} \subset A_{p-1}.$$

More generally, the degree p elements in $B_p = \Omega H_*(\Sigma^2 X)_p$, which are (after desuspension) cycles for d_A, belong to $\overline{E}^p_{p,0}$. On the other hand, from the construction (see [Kad05, section 4]) of the operation

$$\cup_1 : \Omega^2 H_*(\Sigma^2 X) \otimes \Omega^2 H_*(\Sigma^2 X) \to \Omega^2 H_*(\Sigma^2 X),$$

the Gerstenhaber bracket $[-;-]_1 = \cup_1 \circ (1 - \tau)$ is compatible with suspensions, i.e. for all $x, y \in \mathrm{s}^{-2} H^+_*(\Sigma^2 X) \subset \mathrm{s}^{-1}\,\Omega H_*(\Sigma^2 X)$, we have $[x;y]_1 = \mathrm{s}^{-1}[\mathrm{s}\,x;\mathrm{s}\,y]_0$. And similarly for the restriction $\xi_1(x) = \mathrm{s}^{-1}\,\xi_0(\mathrm{s}\,x)$. Then

$$t(ad_0(\mathrm{s}^{-1}x_1)\cdots ad_0(\mathrm{s}^{-1}x_{k-1})(\mathrm{s}^{-1}x_k)) = ad_1(x_1)\cdots ad_1(x_{k-1})(x_k)$$
$$t(\xi_0^k(\mathrm{s}^{-1}x)) = \xi_1^k(x),$$

for all $\mathrm{s}^{-1}x, \mathrm{s}^{-1}x_1, ..., \mathrm{s}^{-1}x_k \in \mathrm{s}^{-1} H^+_*(\Sigma^2 X)$. Finally, since Φ^2 is an algebra morphism, it commutes with the differentials. Therefore Φ is a morphism of spectral sequences.

The morphisms Φ^∞ and $T(\iota)$ are clearly isomorphisms. This ends the proof. $\square$

5 The homology of the double cobar construction of a double suspension over $\mathbb{Q}$

In this section the coefficient field is $\mathbb{Q}$.

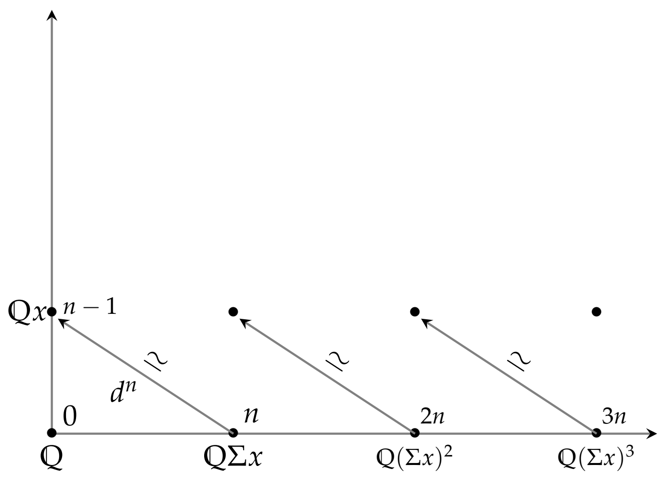
5.1 Free Gerstenhaber algebra structure

Theorem 5.1. *Let the coefficient field be $\mathbb{Q}$. Then $\Omega^2 C_*(\Sigma^2 X)$ has a Connes-Moscovici structure. The induced Gerstenhaber algebra structure on $H_*(\Omega^2 C_*(\Sigma^2 X))$ is free on the reduced homology $H_*^+(X)$.*

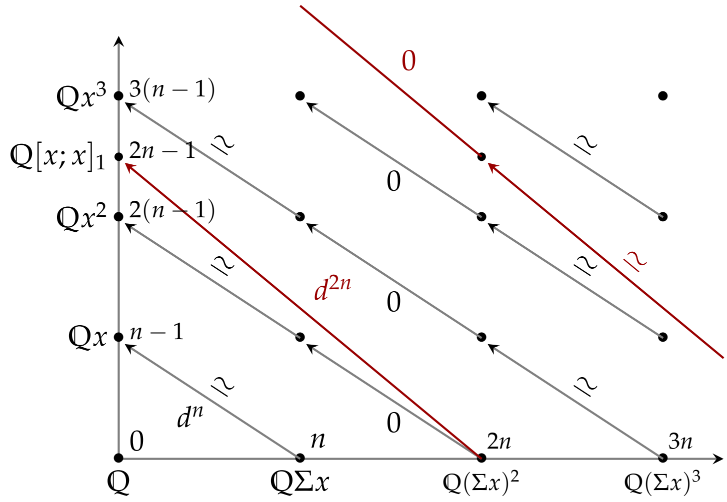
It was proved in [Coh76] that $H_*(\Omega^2\Sigma^2 Y)$ is the free Gerstenhaber algebra on $H_*^+(Y)$, then we obtain

Corollary 5.2. *The Gerstenhaber algebras $H_*(\Omega^2 C_*(\Sigma^2 X))$ and $H_*(\Omega^2|\Sigma^2 X|)$ are isomorphic.*

Proof of Theorem 5.1. The proof is analogous to that of Theorem 4.1. We replace the functor L_{1r} by L_1 adjoint of the forgetful functor from the category of degree 1 Lie algebras to the category of graded vector spaces. The symmetric algebra $S(V)$ on a graded vector space V, is TV/I where I is the ideal generated by the elements $v \otimes w - (-1)^{|v||w|} w \otimes v$. Then $S(L_1(V))$ is a Gerstenhaber algebra; its bracket is $[v,w]_1 = vw - (-1)^{(|v|-1)(|w|-1)} wv$ for $v, w \in V$. The main difference is the decomposition of the model spectral sequence E. Indeed, on a generator $\Sigma x \in H_n^+(\Sigma X)$ of even degree n, the spectral sequence is:



On a generator $\Sigma x \in H_n^+(\Sigma X)$ of odd degree n, the spectral sequence is:



The spectral sequence E decomposes into elementary spectral sequences

$$P(y) \otimes S(t(y)) \text{ and } S(z) \otimes P(t(z))$$

where y and z run through the respectively even and odd degree $[-;-]_0$-elementary products of $TH_*^+(\Sigma X)$. □

5.2 BV-algebra structure

5.2.1 Free Batalin-Vilkovisky algebra

We fix a connected vector space V and an operator $\Delta : V_* \to V_{*+1}$ such that $\Delta^2 = 0$. We define the free Batalin-Vilkovisky algebra on (V, Δ) as

$$S(L_1(V \oplus \Delta(V))).$$

The operator Δ is extended as BV-operator via the formula

$$\Delta(xy) = \Delta(x)y + (-1)^{|x|}x\Delta(y) + (-1)^{|x|}[x;y]_1,$$

for all homogeneous elements $x, y \in L_1(V)$, and

$$\Delta([x;y]_1) = [\Delta(x);y]_1 + (-1)^{|x|-1}[x;\Delta(y)]_1,$$

for all homogeneous elements $x, y \in L_1(V)$.
If $\Delta : V \to V$ is trivial then $\Delta : S(L_1(V)) \to S(L_1(V))$ is the BV-operator given by the extension of the Gerstenhaber bracket. The resulting Batalin-Vilkovisky structure is called *canonical Batalin-Vilkovisky algebra structure* $S(L_1(V))$, cf. [TT00, Section 1.1].
Moreover, if the free Gerstenhaber algebra $S(L_1(V))$ is a Batalin-Vilkovisky algebra such that the BV-operator is zero when it is restricted to V, then $S(L_1(V))$ is the canonical Batalin-Vilkovisky algebra.

5.2.2 Batalin-Vilkovisky algebra on the homology of a double loop space

Let us make explicit the Batalin-Vilkovisky structure given by E. Getzler [Get94] on $H_*(\Omega^2\Sigma^2 X)$.

Let Y be a S^1-space, that is a pointed topological space with base point $*$, endowed with an action of the circle S^1 such that $g \cdot * = *$ for all $g \in S^1$. E. Getzler shows [Get94] that the diagonal action of S^1 on $\Omega^2 Y = \mathrm{Hom}(S^2, Y)$:

$$\Delta : S^1 \times \mathrm{Hom}(S^2, Y) \to \mathrm{Hom}(S^2, Y)$$
$$(g, \phi) \mapsto (g \cdot \phi)(s) := g \cdot \phi(g^{-1} \cdot s),$$

induces a Batalin-Vilkovisky algebra structure on $H_*(\Omega^2 Y)$. For a double suspension $Y = \Sigma^2 X = S^2 \wedge X$, we endowed Y with the canonical action of S^1 on S^2 and the trivial action on X i.e. $g \cdot (s \wedge x) = (g \cdot s) \wedge x$ for all $g \in S^1$ et $s \wedge x \in S^2 \wedge X$. Then the adjunction $X \to \Omega^2\Sigma^2 X$ is equivariant for the trivial action on X and the diagonal action on $\Omega^2\Sigma^2 X$. As a consequence, cf. [GM08], $(H_*(\Omega^2\Sigma^2 X), \Delta)$ is a canonical Batalin-Vilkovisky algebra, free on $H_*^+(X)$.

5.2.3 Homology of double cobar construction

Theorem 5.3. *Let the coefficient field be $\mathbb{Q}$. Then $\Omega^2 C_*(\Sigma^2 X)$ has a Connes-Moscovici structure. The induced BV-algebra structure on $(H_*(\Omega^2 C_*(\Sigma^2 X)), \Delta_{CM})$ is free on the reduced homology $H_*^+(X)$.*

As a consequence,

Corollary 5.4. *The Batalin-Vilkovisky algebras $(H_*(\Omega^2 C_*(\Sigma^2 X)), \Delta_{CM})$ and $(H_*(\Omega^2|\Sigma^2 X|), \Delta)$ are isomorphic.*

Proof of Theorem 5.3. From Proposition 1.3 and the formality Theorem 2.2, $\Omega^2 H_*(\Sigma^2 X)$ has a Connes-Moscovici structure and its homology $H_*(\Omega^2 H_*(\Sigma^2 X))$ is a Batalin-Vilkovisky algebra isomorphic to $H_*(\Omega^2 C_*(\Sigma^2 X))$.
On the other hand, the simplicial set X is seen as endowed with a trivial action of S^1. Then the inclusion

$$\iota : (H_*^+(X), 0) \to H_*(\Omega^2 H_*(\Sigma^2 X)),$$

lifts to a unique morphism of Batalin-Vilkovisky algebras,

$$SL(\iota) : S(L_1(H_*^+(X))) \to H_*(\Omega^2 H_*(\Sigma^2 X)),$$

with the usual commutative diagram.

Let us observe the operator

$$H(\Delta_{CM}) : H_*(\Omega^2 H_*(\Sigma^2 X)) \to H_*(\Omega^2 H_*(\Sigma^2 X)),$$

induced by the Connes-Moscovici operator $\Delta_{CM} : \Omega^2 H_*(\Sigma^2 X) \to \Omega^2 H_*(\Sigma^2 X)$. By construction we have (see [CM00, (2.22)]):

$$\Delta_{CM}|_{s^{-1}\Omega H_*(\Sigma^2 X)} = 0 \quad \text{and } \textit{a fortiori} \quad \Delta_{CM}|_{s^{-2} H_*^+(\Sigma^2 X)} = 0.$$

Then

$$H(\Delta_{CM})|_{H_*^+(X)} = 0.$$

Moreover, from Theorem 5.1, the underlying Gerstenhaber algebra $H_*(\Omega^2 H_*(\Sigma^2 X))$ is the Gerstenhaber algebra, free on $H_*^+(X)$. This concludes the proof by the argument from section 5.2.1. $\square$

6 Appendix

In this appendix we write down the whole relations among the co-operations defining a Hirsch coalgebra. Homotopy G-coalgebras are particular Hirsch coalgebras.

Definition 6.1. A Hirsch coalgebra $(C, d, \nabla_C, E = \{E^{i,j}\})$ is the data of a coproduct $\nabla_E : \Omega C \to \Omega C \otimes \Omega C$ on the cobar construction of the 1-connected dg-coalgebra (C, d, ∇_C) such that ΩC is a co-unital dg-bialgebra.

The coproduct ∇_E corresponds to a twisting cochain

$$E : s^{-1} C^+ \to \Omega C \otimes \Omega C.$$

The (i, j)-component is

$$E^{i,j} : s^{-1} C^+ \to (s^{-1} C^+)^{\otimes i} \otimes (s^{-1} C^+)^{\otimes j}.$$

To alleviate notation, we set $\overline{C} := s^{-1} C^+$.
Thus

$$E^{i,j} : \overline{C} \to \overline{C}^{\otimes i} \otimes \overline{C}^{\otimes j}.$$

The degree of $E^{i,j}$ is 0.

Let us make explicit the co-unit condition, the compatibility with the differential and the coassociativity of the coproduct.

6.0.4 The co-unit condition

The following relation $(\epsilon \otimes 1)\nabla_E = (1 \otimes \epsilon)\nabla_E = Id$ gives:

$$E^{0,1} = E^{1,0} = Id_{\overline{C}} \ \text{ and } \ E^{0,k} = E^{k,0} = 0 \ \text{ for } k > 1. \tag{6.1}$$

6.0.5 The compatibility with the differential

We investigate the equation $(d_\Omega \otimes 1 + 1 \otimes d_\Omega)\nabla_E = \nabla_E d_\Omega$.

On the component $\overline{C}^{\otimes k} \otimes \overline{C}^{\otimes l} \subset \Omega C \otimes \Omega C$, we obtain:

$$\begin{aligned} d_{\overline{C}^{\otimes k} \otimes \overline{C}^{\otimes l}} E^{k,l} - E^{k,l} d_{\overline{C}} = & - \sum_{i+2+j=k} ((1^{\otimes i} \otimes \nabla_{\overline{C}} \otimes 1^{\otimes j}) \otimes 1^{\otimes l}) E^{k-1,l} \\ & - \sum_{i+2+j=l} (1^{\otimes k} \otimes (1^{\otimes i} \otimes \nabla_{\overline{C}} \otimes 1^{\otimes j})) E^{k,l-1} \\ & + \sum_{\substack{k_1+k_2=k \\ l_1+l_2=l}} \mu_{\Omega \otimes \Omega} (E^{k_1,l_1} \otimes E^{k_2,l_2}) \nabla_{\overline{C}}, \end{aligned} \tag{6.2}$$

where $k_1 + l_1 > 0$ and $k_2 + l_2 > 0$.

The terms $\mu_{\Omega\otimes\Omega}(E^{k_1,l_1} \otimes E^{k_2,l_2})\nabla_{\overline{C}}$ read as follows. We set

$$E^{k,l}(a) = a^{|\overleftarrow{k}} \otimes a^{|\overrightarrow{l}} \quad \text{and} \quad \nabla_{\overline{C}}(a) = a^1 \otimes a^2.$$

Then

$$\mu_{\Omega\otimes\Omega}(E^{k_1,l_1} \otimes E^{k_2,l_2})\nabla_{\overline{C}} = \pm(a^{1|\overleftarrow{k_1}} \otimes a^{2|\overleftarrow{k_2}}) \otimes (a^{1|\overrightarrow{l_1}} \otimes a^{2|\overrightarrow{l_2}}).$$

On the component $\overline{C} \otimes \overline{C} \subset \Omega C \otimes \Omega C$ the equation (6.2) gives:

$$d_{\overline{C}\otimes\overline{C}} E^{1,1} - E^{1,1} d_{\overline{C}} = \nabla_{\overline{C}} + \tau \nabla_{\overline{C}}; \tag{6.3}$$

that is, on $C^+ \otimes C^+$:

$$d_{C^+\otimes C^+} E_+^{1,1} + E_+^{1,1} d_{C^+} = \nabla_{C^+} - \tau \nabla_{C^+}, \tag{6.4}$$

where $E^{1,1} = (\mathrm{s}^{-1} \otimes \mathrm{s}^{-1}) E_+^{1,1}\, \mathrm{s}$.
Thus $E_+^{1,1}$ is a chain homotopy for the cocommutativity of the coproduct ∇_{C^+}. We distinguish it from the co-operations $E^{i,j}$ and we denote it by ∇_1 as a dual analogue to the $\cup_1$-product.

6.0.6 The coassociativity condition

On the component $\overline{C}^{\otimes i} \otimes \overline{C}^{\otimes j} \otimes \overline{C}^{\otimes k}$ we obtain:

$$\begin{aligned} &\sum_{n=0}^{i+j} \sum_{\substack{i_1+\ldots+i_n=i \\ j_1+\ldots+j_n=j}} (\mu^{(n)}_{\Omega\otimes\Omega}(E^{i_1,j_1} \otimes \ldots \otimes E^{i_n,j_n}) \otimes 1^{\otimes k}) E^{n,k} \\ = &\sum_{n=0}^{j+k} \sum_{\substack{j_1+\ldots+j_n=j \\ k_1+\ldots+k_n=k}} (1^{\otimes i} \otimes \mu^{(n)}_{\Omega\otimes\Omega}(E^{j_1,k_1} \otimes \ldots \otimes E^{j_n,k_n})) E^{i,n}. \end{aligned} \tag{6.5}$$

In particular, for $i = j = k = 1$, we have:

$$(E^{1,1} \otimes 1)E^{1,1} + (1 \otimes 1)E^{2,1} + (\tau \otimes 1)E^{2,1} = (1 \otimes E^{1,1})E^{1,1} + (1 \otimes 1)E^{1,2} + (1 \otimes \tau)E^{1,2}.$$

That is,

$$(\nabla_1 \otimes 1)\nabla_1 - (1 \otimes \nabla_1)\nabla_1 = (1 \otimes (1+\tau))E^{1,2} - ((1+\tau) \otimes 1)E^{2,1}. \tag{6.6}$$

The lack of coassociativity of ∇_1 is controlled by the co-operations $E^{1,2}$ and $E^{2,1}$.

Definition 6.2. A homotopy G-coalgebra $(C, d, \nabla_C, E^{i,1})$ is a Hirsch coalgebra such that $E^{i,j} = 0$ for $i \geq 2$.

Proposition 6.3. *Let $(C, d, \nabla_C, E^{i,j})$ be a Hirsch coalgebra. Then $E^{i,j} = 0$ for $i \geq 2$ is equivalent to the following left-sided condition:*
(6.7)
$\forall r \in \mathbb{N},\ J_r := \oplus_{n \leq r} \overline{C}^{\otimes n}$ *is a left co-ideal for the coproduct* $\nabla_E : \Omega C \to \Omega C \otimes \Omega C$.

Proof. If J_r is a left co-ideal then $\nabla_E(J_r) \subset J_r \otimes \Omega C$. Then for $r = 1$, we have $\nabla_E(\overline{C}) = \sum_{i,j} E^{i,j}(\overline{C}) \subset \overline{C} \otimes \Omega C$. This implies that $E^{i,j} = 0$ for $i \geq 2$. Conversely, if $E^{i,j} = 0$ for $i \geq 1$, then

$$\nabla_E(c_1 \otimes \cdots \otimes c_n) = \sum_{i_1, \cdots, i_n, j_1, \cdots, j_n} \mu_\Omega^{(n)}(E^{i_1,j_1} \otimes \cdots \otimes E^{i_n,j_n})(c_1 \otimes \cdots c_n)$$

belongs to $\oplus \overline{C}^{i_1 + \cdots + i_n} \otimes \Omega C$. Since $i_s \leq 1$ for $1 \leq s \leq n$, then $i_1 + \cdots + i_n \leq n$. □

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